Fundamental Solutions for the Tricomi Operator, II

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Abstract

In this paper we explicitly calculate fundamental solutions for the Tricomi operator, relative to an arbitrary point in the plane, and show that all such fundamental solutions originate from the hypergeometric function $F(1/6, 1/6; 1; \zeta)$ that is obtained when we look for homogeneous solutions to the reduced hyperbolic Tricomi equation.

1 Introduction

The Tricomi operator

$$\mathcal{T} = y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1.1}$$

one of the simplest examples of a partial differential operator of *mixed type*, is: (i) *elliptic* in the upper half plane ($y > 0$); *parabolic* along the x-axis ($y = 0$); and (iii) *hyperbolic* in the lower half plane ($y < 0$).

Our aim is to obtain explicit solutions in the sense of distributions or generalized functions of the equation

$$\mathcal{T}E = \delta(x - x_0, y - y_0), \tag{1.2}$$

where $\delta(x - x_0, y - y_0)$ is the Dirac function at (x_0, y_0) an arbitrary point in the plane. A solution E of (1.2) is said to be a *fundamental solution relative to the point* (x_0, y_0).

In a previous paper [2] we considered the case when $x_0 = y_0 = 0$ and proved the existence of two remarkable fundamental solutions that clearly reflect the fact that the operator changes type across the x-axis. In this paper we study the general case when $x_0 = y_0 \neq 0$ and compare our results to those of [2].

For sake of completeness and in order to make the reading of this paper independent of that of [2], we briefly review the contents of that paper. It is known that the equation $9x^2 + 4y^3 = 0$ defines the two characteristics for the Tricomi operator that emanate from the origin. These characteristics, which are tangent to the y-axis at the origin, divide the plane in two disjoint regions D_+ and D_- (Figure 1) defined as follows:

$$D_+ = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 > 0\},$$

the region "outside" the characteristics, and

$$D_- = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 < 0\},$$

the region "inside" the characteristics.

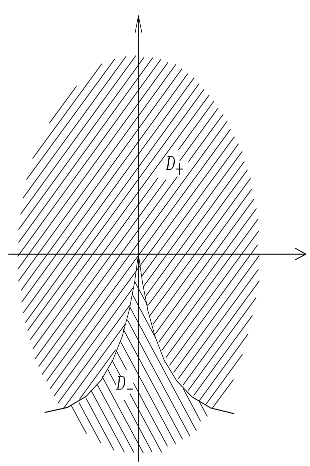

Figure 1

The first fundamental solution is defined by

$$F_+(x, y) = \begin{cases} C_+(9x^2 + 4y^3)^{-1/6} & \text{in } D_+ \\ 0 & \text{elsewhere} \end{cases} \tag{1.3}$$

with

$$C_+ = -\frac{1}{2^{1/3}\cdot 3^{1/2}} F(\frac{1}{6},\frac{1}{6};1;1), \tag{1.4}$$

and the second one is defined by

$$F_-(x,y) = \begin{cases} C_-|9x^2+4y^3|^{-1/6} & \text{in } D_- \\ \\ 0 & \text{elsewhere} \end{cases} \tag{1.5}$$

with

$$C_- = \frac{1}{2^{1/3}} F(\frac{1}{6},\frac{1}{6};1;1). \tag{1.6}$$

In the expressions of both C_+ and C_-, the constant

$$F(\frac{1}{6},\frac{1}{6};1;1) = \frac{\Gamma(2/3)}{\Gamma^2(5/6)}$$

is the value of the hypergeometric function $F(1/6,1/6;1;\zeta)$ at $\zeta = 1$.

We remark that in [2] the constants C_+ and C_- were respectively denoted by

$$-\frac{\Gamma(1/6)}{3\cdot 2^{2/3}\pi^{1/2}\Gamma(2/3)} \qquad \text{and} \qquad \frac{3\Gamma(4/3)}{2^{2/3}\pi^{1/2}\Gamma(5/6)}.$$

At that time, we were unaware of the role played by the hypergeometric function $F(1/6,1/6;1;\zeta)$ in the study of fundamental solutions for the Tricomi operator.

We also observe that according to formula (1.5), a perturbation at the origin spreads out to the entire region D_-, inside the characteristics. On the other hand, according to formula (1.3), the same perturbation spreads out to the whole elliptic region ($y > 0$) and also to the hyperbolic region ($y < 0$) outside the characteristics. Since the Tricomi operator is invariant under translations along the x-axis, the same phenomena take place for fundamental solutions that correspond to the Dirac measure δ concentrated at an arbitrary point $(a,0)$ on the x-axis.

It is well know that the Tricomi operator describes the transition from subsonic flow (elliptic region) to supersonic flow (hyperbolic region). In the extensive literature on the Tricomi operator one finds in the works of several authors, among others, Agmon [1], Friedrichs [6], Gelfand [unpublished], Germain and Bader [8], Landau and Lifshitz [9], Leray [11], and Morawetz [12], a large number of examples of solutions to different problems that show

the interaction between these two regions. It seems that the two distributions $F_+(x,y)$ and $F_-(x,y)$ are the simplest of such examples.

We now return to equation (1.2). In view of the invariance of the Tricomi operator under translations parallel to the x-axis, the problem of solving that equation is equivalent to that of solving

$$\mathcal{T}E = \delta(x, y-b), \tag{1.7}$$

where b is an arbitrary real number and $\delta(x, y-b)$ denotes the Dirac measure concentrated at the point $(0, b)$.

Consider the case $b < 0$, that is, the point $(0, b)$ is situated in the hyperbolic region. We introduce the *characteristic coordinates*

$$\ell = 3x + 2(-y)^{3/2} \quad \text{and} \quad m = 3x - 2(-y)^{3/2}, \tag{1.8}$$

set $\ell_0 = 2(-b)^{3/2}$, and to let $a > 0$ be such that $3a = 2(-b)^{3/2}$. Note that $(\ell_0, -\ell_0)$ represents the point $(0, b)$ in characteristic coordinates. As shown

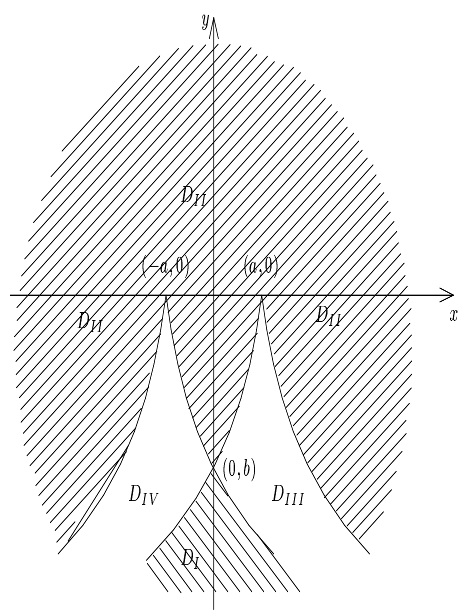

Figure 2

in Figure 2, two characteristics with equations

$$3(x-a) + 2(-y)^{3/2} = 0 \quad \text{and} \quad 3(x+a) - 2(-y)^{3/2} = 0 \tag{1.9}$$

pass through the point $(0, b)$. In characteristic coordinates, these are the two half-lines

$$(1.10) \qquad \ell = \ell_0, \ -\infty < m \leq \ell_0, \quad \text{and} \quad m = -\ell_0, \ -\ell_0 \leq \ell < +\infty$$

that originate from the point $(\ell_0, -\ell_0)$.

The characteristics (1.9) meet the x-axis, respectively, at the points $(a, 0)$ and $(-a, 0)$. Two new characteristics originate from these two points, namely,

$$(1.11) \qquad 3(x-a) - 2(-y)^{3/2} = 0 \quad \text{and} \quad 3(x+a) + 2(-y)^{3/2} = 0,$$

which we call *reflected characteristics.* The corresponding equations in characteristic coordinates are

$$(1.12) \qquad m = \ell_0, \ \ell_0 \leq \ell < +\infty \quad \text{and} \quad \ell = -\ell_0, \ -\infty < m \leq -\ell_0.$$

The two characteristics through the point $(0, b)$ and the reflected characteristics divide the plane into four disjoint regions denoted by D_I, D_{II}, D_{III}, and D_{IV}, and illustrated in Figure 2.

In this paper, we show existence of four fundamental solutions each supported by the closure of the corresponding region. Of these solutions, only two have physical meaning: the one defined in the region D_I and the other defined in the region D_{II}. This situation is similar to what happens in the case of the wave operator in two dimensions where the two relevant fundamental solutions are the ones supported by the forward and backward light-cones. As we will see, it is the hypergeometric function $F(1/6, 1/6; 1; \zeta)$ that plays a crucial role in defining these four fundamental solutions.

The plan of this paper is as follows. We prove in Section 2 that the function

$$(1.13) \ E(\ell, m; \ell_0, -\ell_0) = (\ell+\ell_0)^{-1/6}(\ell_0-m)^{-1/6} F(\frac{1}{6}, \frac{1}{6}; 1; \frac{(\ell-\ell_0)(m+\ell_0)}{(\ell+\ell_0)(m-\ell_0)})$$

is a solution of $\mathcal{T}_h u = 0$, where $\mathcal{T}_h$ denotes the *reduced hyperbolic* Tricomi equation (2.1).

After replacing into (1.13), ℓ and m by their expressions in (1.8), we obtain the following function of x and y:

$$\begin{aligned}(1.14) \quad E(x, y; 0, b) \ &= \ e^{i\pi/6}[9(x^2-a^2) + 4y^3 - 12a(-y)^{3/2}]^{-1/6} \times \\ &\times F\left(\frac{1}{6}, \frac{1}{6}; 1; \frac{9(x^2-a^2) + 4y^3 + 12a(-y)^{3/2}}{9(x^2-a^2) + 4y^3 - 12a(-y)^{3/2}}\right).\end{aligned}$$

This is the function that generates the four fundamental solutions relative to the point $(0, b)$. In Section 3 we define the distribution $E_I(x, y; 0, b)$ as the restriction of $E(x, y; 0, b)$, after multiplication by a suitable constant, to the region D_I and show that E_I is a fundamental solution of the Tricomi operator. The region D_I is entirely contained in the hyperbolic region where it is natural to use characteristic coordinates. Theorem 3.1 is then proved, via integration by parts and by using the results of Proposition 2.1. As a consequence of it we show that, as $(0, b)$ tends to $(0, 0)$, $E_I(x, y; 0, b)$ tends, in the sense of distributions, to $F_-(x, y)$, the fundamental solution defined by (1.5).

The fundamental solutions supported by the closure of the regions D_{III} and D_{IV} are defined and studied in Section 4. In both regions, it is necessary to take into account the singularities of $E(x, y; 0, b)$ along the reflected characteristics. The results of Proposition 4.1, that describe the asymptotic behavior of the hypergeometric function $F(1/6, 1/6; 1; \zeta)$ as $|\zeta| \to +\infty$, are then needed in the proof of Theorem 4.1. We note that as $b \to 0$, both fundamental solutions E_{III} and E_{IV} tend to 0.

In Section 5, we define $E_{II}(x, y; 0, b)$ as the restriction of $E(x, y; 0, b)$, after multiplication by a suitable constant, to D_{II} and show that $\mathcal{T} E_{II} = \delta(0, b)$. Since E_{II} is complex valued and the Tricomi operator has real coefficients, both its complex conjugate and real part are also fundamental solutions relative to the point $(0, b)$. The imaginary part of E_{II} is then a solution of the homogeneous equation $\mathcal{T} u = 0$ and we will call it a *Tricomi harmonic function*.

Contrary to what was proved in Corollary 3.1, it is not true that the fundamental solution $F_+(x, y)$ defined by (1.3) is the limit of $E_{II}(x, y; 0, b)$, or its real part, as $(0, b) \to (0, 0)$. It is necessary to take a suitable linear combination of E_{II} and its complex conjugate in order to achieve this result.

As a final remark, we mention that in his paper [11], J. Leray described a general method, based upon the theory of analytic functions of several complex variables to find fundamental solutions for a class of hyperbolic linear differential operator with analytic coefficients. In particular, he showed how his method could be used to obtain, in the hyperbolic region, a fundamental solution for the Tricomi operator relative to a point $(0, b)$. He also produced an explicit formula for the fundamental solution in terms of the hypergeometric function $F(1/6, 1/6; 1; \zeta)$. Our method is simpler, more direct, and gives us global fundamental solutions that clearly reflect the change of type of the Tricomi operator across the x-axis.

We would like to thank Abbas Bahri, Fernando Cardoso and Vladimir Retakh for several helpful discussions.

2 A Special Solution to $\mathcal{T}_h u = 0$

In characteristic coordinates (1.8) the Tricomi operator $\mathcal{T}$ becomes

$$\mathcal{T}_h = \frac{\partial^2}{\partial\ell\partial m} - \frac{1/6}{\ell - m}(\frac{\partial}{\partial\ell} - \frac{\partial}{\partial m}), \tag{2.1}$$

and we call $\mathcal{T}_h$ the *reduced hyperbolic* form of $\mathcal{T}$.

We now look for homogeneous solutions of the equation $\mathcal{T}_h u = 0$. Every homogeneous function of ℓ and m of degree λ, a complex number, can be written as

$$u(\ell, m) = \ell^\lambda \phi(t),$$

where ϕ is a function of a single variable $t = m/\ell$. Direct substitution into (2.1) shows that $\phi(t)$ must be a solution of the hypergeometric equation

$$t(1-t)\phi''(t) + [(\frac{5}{6} - \lambda) - (\frac{7}{6} - \lambda)t]\phi'(t) + \frac{\lambda}{6}\phi(t) = 0.$$

As a solution of this equation, we choose the following hypergeometric function $F(-\lambda, 1/6; 5/6 - \lambda; t)$ extended, by analytic continuation, to the whole complex plane $\mathbb{C}$ minus the cut $[1, \infty)$. Since we are looking for fundamental solutions to the Tricomi operator we take for λ the value $-1/6$, as previously indicated in our joint paper [2]. Thus

$$u(\ell, m) = \ell^{-1/6} F(\frac{1}{6}, \frac{1}{6}; 1; \frac{m}{\ell})$$

is a solution of $\mathcal{T}_h u = 0$.

Let now (ℓ_0, m_0) be an arbitrary point in $\mathbb{R}^2$ and consider the change of variables

$$\ell \to \frac{\ell - m_0}{\ell - \ell_0} \qquad m \to \frac{m - m_0}{m - \ell_0}.$$

After unenlightening calculations, one can show that the function

$$E(\ell, m; \ell_0, m_0) = (\ell - m_0)^{-1/6}(\ell_0 - m)^{-1/6} F(\frac{1}{6}, \frac{1}{6}; 1; \frac{(\ell - \ell_0)(m - m_0)}{(\ell - m_0)(m - \ell_0)}) \tag{2.2}$$

is also a solution of equation (2.1). This is the special solution to $\mathcal{T}_h u = 0$ that we are looking for.

Consider now the adjoint to equation (2.1):

$$\mathcal{T}_h^* v = \frac{\partial^2 v}{\partial \ell \partial m} + \frac{1/6}{l-m}(\frac{\partial v}{\partial \ell} - \frac{\partial v}{\partial m}) - \frac{1/3}{(\ell-m)^2} v = 0. \tag{2.3}$$

One can see that if v is a solution of equation (2.3), then $u = (\ell - m)^{-1/3} v$ is a solution of equation (2.1). Since $E(\ell, m; \ell_0, m_0)$ is a solution of (1.13), it follows that the function

$$R(\ell, m; \ell_0, m_0) = (\ell - m)^{1/3} E(\ell, m; \ell_0, m_0) \tag{2.4}$$

is a solution of the adjoint equation (2.3).

Proposition 2.1. $R(\ell, m; \ell_0, m_0)$ *is the unique solution of* $\mathcal{T}_h^* v = 0$ *that satisfies the following conditions:*

(i) $R_\ell = \dfrac{1/6}{\ell - m} R$ *along the line* $m = m_0$,

(ii) $R_m = \dfrac{-1/6}{\ell - m} R$ *along the line* $\ell = \ell_0$;

(iii) $R(\ell_0, m_0; \ell_0, m_0) = 1$.

Proof. Clearly, conditions (i), (ii), and (iii) imply uniqueness for R. As $m = m_0$, the argument of the hypergeometric function in (2.2) equals zero and so $F(1/6, 1/6; 1; 0) = 1$. Thus, along the line $m = m_0$, we have that

$$R(\ell, m_0; \ell_0, m_0) = \left(\frac{\ell - m_0}{\ell_0 - m_0}\right)^{1/6} = e^{\displaystyle\int_{\ell_0}^{\ell} a(t)\, dt}, \tag{2.5}$$

where $a(t) = \dfrac{1/6}{t - m_0}$. Therefore $R_\ell = \dfrac{1/6}{\ell - m} R$ along $m = m_0$. In the same manner, one can see that $R_m = \dfrac{-1/6}{\ell - m} R$ along the line $\ell = \ell_0$. Finally, it is clear that $R(\ell_0, m_0; \ell_0, m_0) = 1$. □

Remark 1. The function $R(\ell, m; \ell_0, m_0)$ is the Riemann function of the operator $\mathcal{T}_h$ relative to the point (ℓ_0, m_0), (see [3]).

If we go back to the notations in the Introduction, with $(0,b)$ and $b<0$, and let $\ell_0 = 2(-b)^{3/2}$, then $m_0 = -\ell_0$ and we obtain from equation (2.2) the equation (1.13) that, for further reference, we write as follows:

$$E(\ell, m; \ell_0, -\ell_0) = (\ell+\ell_0)^{-1/6}(\ell_0 - m)^{-1/6} F(\frac{1}{6}, \frac{1}{6}; 1; \zeta) \tag{2.6}$$

where

$$\zeta = \frac{(\ell-\ell_0)(m+\ell_0)}{(\ell+\ell_0)(m-\ell_0)}. \tag{2.7}$$

Similarly, after replacing ℓ and m by their expressions in (1.8) we obtain the function (1.14) that we rewrite as

$$E(x,y;0,b) = e^{i\pi/6}[9(x^2-a^2)+4y^3-12a(-y)^{3/2}]^{-1/6} \times F\left(\frac{1}{6}, \frac{1}{6}; 1; \zeta\right) \tag{2.8}$$

where

$$\zeta = \frac{9(x^2-a^2)+4y^3+12a(-y)^{3/2}}{9(x^2-a^2)+4y^3-12a(-y)^{3/2}}. \tag{2.9}$$

We wish to analyze this function. Denote by r_a the reflected characteristic $3(x-a) - 2(-y)^{3/2} = 0$ at $(a,0)$ and by r_{-a} the reflected characteristic $3(x+a) + 2(-y)^{3/2} = 0$ at $(-a,0)$.

Proposition 2.2. *$E(x,y;0,b)$ is a real analytic function defined in $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$. The singularities of $E(x,y;0,b)$ occur along the reflected characteristics r_a and r_{-a}.*

Proof. 1. Let

$$z = [3(x+a)+2(-y)^{3/2}][3(x-a)-2(-y)^{3/2}] = 9(x^2-a^2)+4y^3-12a(-y)^{3/2}$$

be the denominator of ζ in the expression (2.9). For $y \le 0$, $z=0$ on $r_a \cup r_{-a}$, it is *negative* in the region inside these characteristics, and it is *positive* in the region outside them. For $y>0$, z is a complex number.

It is possible to choose the argument of z so that it varies continuously in the region $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$. Indeed, define $\phi : \mathbb{R}^2 \backslash (r_a \cup r_{-a}) \to S_1$, the unit circumference, by

$$\phi(x,y) = \frac{9(x^2-a^2)+4y^3-12a(-y)^{3/2}}{\rho},$$

where
$$\rho = |9(x^2 - a^2) + 4y^3 - 12a(-y)^{3/2}|.$$
Since $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$ is contractible, ϕ lifts to $\mathbb{R}$, that is, there exists a continuous function $\theta(x, y)$ defined on $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$ so that
$$9(x^2 - a^2) + 4y^3 - 12a(-y)^{3/2} = \rho e^{-i\theta(x,y)}.$$
We may take $\theta(x, y)$ to be equal zero outside the reflected characteristis and equal π inside them. It follows that
$$z^{-1/6} = [9(x^2 - a^2) + 4y^3 - 12a(-y)^{3/2}]^{-1/6}$$
is well defined and real analytic in $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$. As for
$$F\left(\frac{1}{6}, \frac{1}{6}; 1; \frac{9(x^2 - a^2) + 4y^3 + 12a(-y)^{3/2}}{9(x^2 - a^2) + 4y^3 - 12a(-y)^{3/2}}\right)$$
it is also real analytic in the same region. □

The following remaks are in order:

1) For $y = 0$ we have
$$E(x, 0; 0, b) = F(\frac{1}{6}, \frac{1}{6}, 1; 1)|x^2 - a^2|^{-1/6}, \quad \forall x \in (-a, a)$$
and
$$E(x, 0; 0, b) = e^{i\pi/6} F(\frac{1}{6}, \frac{1}{6}, 1; 1)(x^2 - a^2)^{-1/6}, \quad \forall x \notin (-a, a).$$

2) For $y > 0$ we have
$$\zeta = \frac{\bar{z}}{z} = \rho e^{i2\theta(x,y)}, \quad \text{so} \quad |\zeta| = 1.$$
It is known (see [5]) that the hypergeometric function $F(1/6, 1/6; 1; \zeta)$ is then given by its hypergeometric series and this series is absolutely convergent in the closed disk $|\zeta| \le 1$.

3) Along the characteristics
$$3(x - a) + 2(-y)^{3/2} = 0 \quad \text{and} \quad 3(x + a) - 2(-y)^{3/2} = 0$$

through the point $(0,b)$ the function $E(x,y;0,b)$ is equal to

$$E = 2^{-2/3}(by)^{-1/4}$$

and so it has a singularity of order $-1/4$ at $y=0$.

4) Along both reflected characteristics r_{-a} and r_a, the function $E(x,y;0,b)$ has a logarithmic singularity. This follows from Proposition 4.1, in Section 4, that describes the asymptotic behavior of the hypergeometric function $F(1/6,1/6;1;\zeta)$ as $|\zeta| \to +\infty$.

3 The Fundamental Solution in the region D_I

Let $E(x,y;0,b)$ be the function defined by the expressions (2.8) and (2.9) and define

$$E_I(x,y;0,b) = \begin{cases} \dfrac{1}{2^{1/3}} E(x,y;0,b) & \text{in } D_I \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

Since $E(x,y;0,b)$ is $\mathcal{C}^\infty$ in D_I and bounded on the boundary of D_I, it follows that $E_I(x,y;0,b)$ is a locally integrable function and defines a distribution whose support is the closure of D_I.

Theorem 3.1. *$E_I(x,y;0,b)$ is a fundamental solution for the Tricomi operator $\mathcal{T}$ relative to the point $(0,b)$.*

Proof. We must show that

$$\langle E_I, \mathcal{T}\varphi \rangle = \varphi(0,b), \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2_-).$$

Since $E(x,y;0,b)$ is locally integrable in $\mathbb{R}^2$, this is equivalent to showing that

$$\int\int_{\mathbb{R}^2_-} E_I(x,y,;0,b)\mathcal{T}\varphi(x,y)\,dx\,dy = \varphi(0,b), \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^2_-). \tag{3.2}$$

By introducing the characteristic coordinates ℓ and m, noting that in the new variables the Tricomi operator (1.1) becomes

$$y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -2^{2/3}3^2(\ell - m)^{2/3}\left[\frac{\partial^2}{\partial \ell \partial m} - \frac{1/6}{l-m}(\frac{\partial}{\partial \ell} - \frac{\partial}{\partial m})\right],$$

and that the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(\ell,m)} = \frac{1}{2^{1/3}3^2(\ell-m)^{1/3}},$$

we can see that the integral in (3.2) is equal to

$$-\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty}(\ell-m)^{1/3}E(\ell,m;\ell_0,-\ell_0)(\phi_{\ell m}-\frac{1/6}{\ell-m}\phi_\ell+\frac{1/6}{\ell-m}\phi_m)\,d\ell\,dm,$$

where $\phi(\ell,m)$ is $\varphi(x,y)$ in characteristic coordinates. We denote the last integral by I and write it as

$$I = -\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty} R(\ell,m;\ell_0,-\ell_0)(\phi_{\ell m}-\frac{1/6}{\ell-m}\phi_\ell+\frac{1/6}{\ell-m}\phi_m)\,d\ell\,dm, \tag{3.3}$$

after recalling that $(\ell-m)^{1/3}E(\ell,m;\ell_0,-\ell_0) = R(\ell,m;\ell_0,-\ell_0)$.

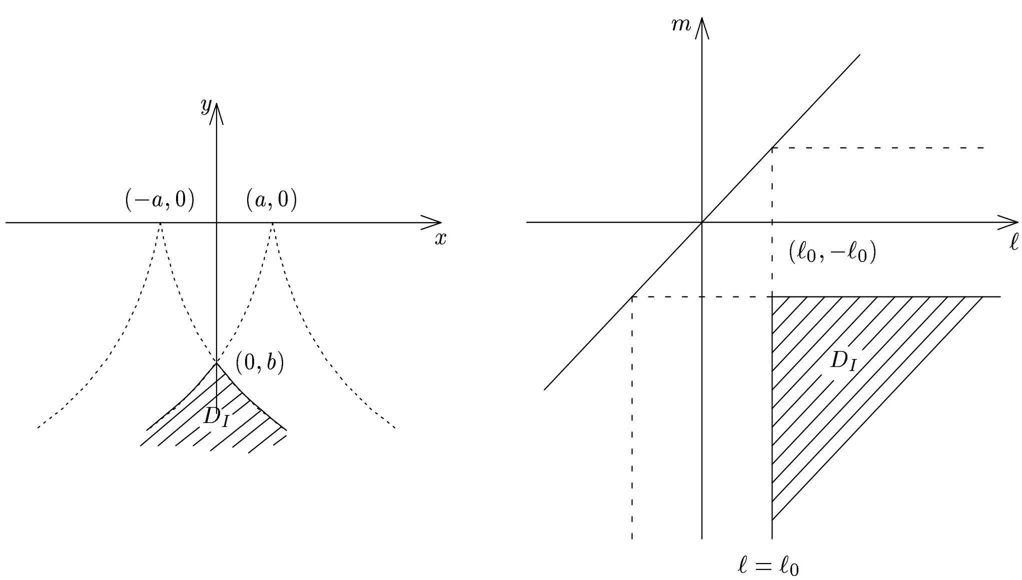

Figure 3

Next we perform several integrations by parts and take into account the properties of the function $R(\ell,m;\ell_0,m_0)$, with $m_0=-\ell_0$, as stated in Proposition 2.1. First consider the term in (3.3) that contains $\phi_{\ell m}$. Integrating by

parts first with respect to m, say, and then with respect to ℓ, one obtains

$$(3.4) \qquad -\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty} R\phi_{\ell m}\, d\ell\, dm = \phi(\ell_0,-\ell_0)$$

$$+\frac{1}{6}\int_{\ell_0}^{\infty}\frac{R(\ell,-\ell_0;\ell_0,-\ell_0)}{\ell+\ell_0}\phi(\ell,-\ell_0)\, d\ell + \frac{1}{6}\int_{-\infty}^{-\ell_0}\frac{R(\ell_0,m;\ell_0,-\ell_0)}{\ell_0-m}\phi(\ell_0,m)\, dm$$

$$-\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty} R_{\ell m}\phi\, d\ell\, dm.$$

Second integrate by parts, relative to ℓ, the term that contains ϕ_ℓ in (3.3) and obtain

$$(3.5) \qquad \frac{1}{6}\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty}\frac{R}{\ell-m}\phi_\ell\, d\ell\, dm =$$

$$-\frac{1}{6}\int_{-\infty}^{-\ell_0}\frac{R(\ell_0,m;\ell_0,-\ell_0)}{\ell_0-m}\phi(\ell_0,m)\, dm - \frac{1}{6}\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty}\frac{\partial}{\partial\ell}\left(\frac{R}{\ell-m}\right)\phi\, d\ell\, dm.$$

Finally integrate by parts, relative to m, the term that contains ϕ_m in (3.3) and get

$$(3.6) \qquad -\frac{1}{6}\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty}\frac{R}{\ell-m}\phi_m\, d\ell\, dm =$$

$$-\frac{1}{6}\int_{\ell_0}^{\infty}\frac{R(\ell,-\ell_0;\ell_0,-\ell_0)}{\ell+\ell_0}\phi(\ell,-\ell_0)\, d\ell + \frac{1}{6}\int_{-\infty}^{-\ell_0}\int_{\ell_0}^{\infty}\frac{\partial}{\partial m}\left(\frac{R}{\ell-m}\right)\phi\, d\ell\, dm.$$

By adding (3.4), (3.5), and (3.6) we obtain that $I = \phi(\ell_0,-\ell_0)$ and this completes the proof. □

As a consequence of Theorem 3.1 we obtain the following result.

Corollary 3.1. *As* $(0,b)\to(0,0)$, *the fundamental solution (3.1) converges in the sense of distributions to* $F_-(x,y)$ *the fundamental solution defined by (1.5) and (1.6).*

Proof. First note that whenever $\mathrm{Re}(c-a-b)>0$, the value $F(a,b;c;1)$ is finite and we have

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Thus $F(1/6,1/6;1;1)$ is well defined. As $(0,b)\to(0,0)$, then $(\ell_0,-\ell_0)\to(0,0)$, and so the limit of $E(\ell,m;\ell_0,-\ell_0)$ defined by (2.6) and (2.7) is

$$(-\ell m)^{-1/6}F(\frac{1}{6},\frac{1}{6};1;1).$$

On the other hand, at the limit the region D_I coincides with D_-, the region inside the two characteristics that originate from the origin, where $\ell m = 9x^2+4y^3<0$. □

Remarks. 1) In our joint paper [2], the multiplicative constant appearing in the definition of $F_-(x,y)$ was given by

$$C_-=\frac{3\Gamma(4/3)}{2^{2/3}\pi^{1/2}\Gamma(5/6)}.$$

It is a matter of verification that this constant coincides with the one given by formula (1.6).

2) Theorem 3.1 could have been proved in a different way, namely, by using the Green's formula (4.5) and by integrating along a suitable contour, as we did in our joint paper [2]. This method which will be explained in the following section, is more adequate in proving existence of the fundamental solutions supported by the closure of the regions D_{II}, D_{III}, and D_{IV}.

4 Fundamental solutions in the regions D_{III} and D_{IV}

For reasons of symmetry it is enough to consider only the region D_{III}. This is the region bounded by part of the characteristic $3(x+a)-2(-y)^{3/2}=0$, part of the characteristic $3(x-a)+2(-y)^{3/2}=0$ and the reflected characteristic $3(x-a)-2(-y)^{3/2}=0$ (See Figure 2).

As we already pointed out, the function $E(x,y;0,b)$ is singular along the reflected characteristic $3(x-a)-2(-y)^{3/2}=0$. The nature of the singularity depends on the behavior of the hypergeometric function $F(1/6,1/6;1;\zeta)$, where $\zeta=(\ell-\ell_0)(m+\ell_0)/(\ell+\ell_0)(m-\ell_0)$, along the characteristic $m=\ell_0$. This is revealed by the following result on the analytic continuation of the hypergeometric series $F(a,a;c;\zeta)$ (see Erdély [5]). The same result also gives us the asymptotic behavior of the analytic extension as $|\zeta|\to\infty$.

Proposition 4.1. *For $\zeta \in \mathbb{C}$ with $|\arg(-\zeta)| < \pi$, we have*

$$F(a, a; c; \zeta) = (-\zeta)^{-a}[\log(-\zeta)u(\zeta) + v(\zeta)] \tag{4.1}$$

where

$$u(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(n!)^2} \zeta^{-n}, \tag{4.2}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$, and

$$v(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(n!)^2} h_n \, \zeta^{-n}, \tag{4.3}$$

with $h_n = 2\Psi(1+n) - \Psi(a+n) - \Psi(c-a-n)$, $\Psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$. Moreover, both series $u(\zeta)$ and $v(\zeta)$ converge for $|\zeta| > 1$.

From this proposition it follows, as we will show in the proof of Theorem 4.1, that the singularity of $E(x, y; 0, b)$ along the reflected characteristics is logarithmic. We may now define

$$E_{III}(x, y; 0, b) = \begin{cases} -\dfrac{1}{2^{1/3}} E(x, y; 0, b) & \text{in } D_{III} \\ \\ 0 & \text{elsewhere,} \end{cases} \tag{4.4}$$

a distribution whose support is the closure of D_{III}. We then have the following result

Theorem 4.1. *$E_{III}(x, y; 0, b)$ is a fundamental solution for the Tricomi operator $\mathcal{T}$ relative to $(0, b)$.*

Proof. 1. We use the following Green's formula for the Tricomi operator (see[2]):

$$\iint_D (E\mathcal{T}\varphi - \varphi\mathcal{T}E)\, dx\, dy = \int_C E(y\varphi_x\, dy - \varphi_y\, dx) - \varphi(yE_x\, dy - E_y\, dx) \tag{4.5}$$

where D is a bounded domain with smooth boundary C. If $\mathcal{T}E = 0$ on D, then last formula becomes

$$\iint_D E\mathcal{T}\varphi\, dx\, dy = \int_C E(y\varphi_x\, dy - \varphi_y\, dx) - \varphi(yE_x\, dy - E_y\, dx) \tag{4.6}$$

and the contour integral may be used to evaluate the double integral. Now throughout this section the domain $D = D_{III}$ lies entirely in the hyperbolic region and so it is more convenient to express the countour integral on the right-hand side of (4.6) in terms of characteristic coordinates.

One can check that, in these coordinates, we have

$$y\varphi_x\,dy - \varphi_y\,dx = A(\ell - m)^{1/3}\psi_\ell\,d\ell - A(\ell - m)^{1/3}\psi_m\,dm,$$

where

$$\psi(\ell, m) = \varphi(\frac{\ell + m}{6}, -(\frac{\ell - m}{4})^{2/3})$$

and $A = 1/2^{2/3}$. Similarly,

$$yE_x\,dy - E_y\,dx = A(\ell - m)^{1/3}E_\ell\,d\ell - A(\ell - m)^{1/3}E_m\,dm.$$

Thus the contour integral in (4.6), denoted by I_C, can be written as

$$I_C = A\int_C (\ell - m)^{1/3}(E\psi_\ell - \psi E_\ell)\,d\ell - A\int_C (\ell - m)^{1/3}(E\psi_m - \psi E_m)\,dm. \tag{4.7}$$

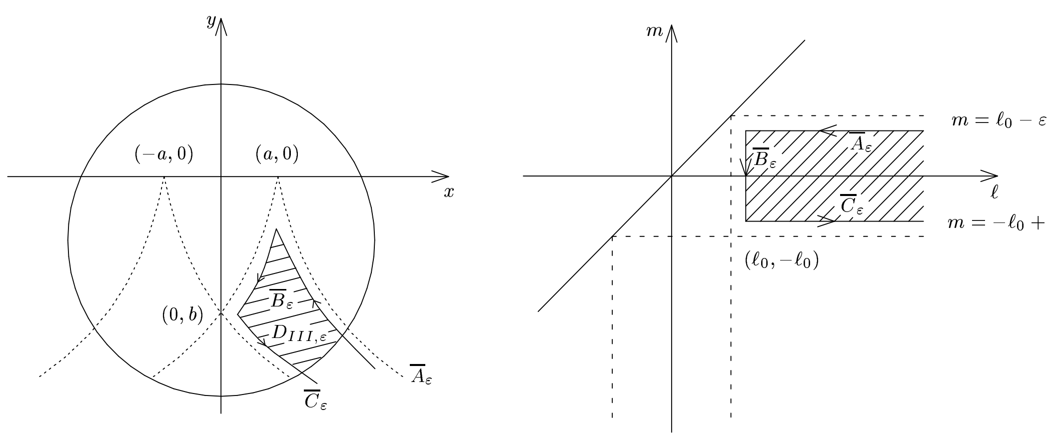

Figure 4

2. We must show that

$$\langle E, \mathcal{T}\varphi\rangle = \lim_{\epsilon\to 0}\iint_{D_\epsilon} E\mathcal{T}\varphi\,dx\,dy = -2^{1/3}\varphi(0,b), \quad \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2), \tag{4.8}$$

where D_ϵ is the intersection of an open disk that contains the support of φ and the region, defined in characteristic coordinates, as follows

$$D_{III,\epsilon} = \{(\ell, m) : \ell > \ell_0 + \epsilon, -\ell_0 + \epsilon < m < \ell_0 - \epsilon\}.$$

Since $\varphi \equiv 0$ on the boundary of the disk, it follows from Green's formula, that we can replace the double integral in (4.8) by an integral of the form (4.7), where the contour C consists of the oriented line segments $\bar{A}_\epsilon$, $\bar{B}_\epsilon$, and $\bar{C}_\epsilon$ shown in Figure 4.

We start by evaluating the simplest of these integrals, namely, the ones along $\bar{B}_\epsilon$ and $\bar{C}_\epsilon$.

3. *The integral along* $\bar{B}_\epsilon$. Denote by $I_{\bar{B}_\epsilon}$ the integral along the line segment, $\ell = \ell_0 + \epsilon$ with m varying from $\ell_0 - \epsilon$ to $-\ell_0 + \epsilon$. Taking into account the orientation of $\bar{B}_\epsilon$, we obtain from formula (4.7) that

$$\begin{aligned}(4.9)\qquad I_{\bar{B}_\epsilon} = {}& A\int_{-\ell_0+\epsilon}^{\ell_0-\epsilon} (\ell_0 + \epsilon - m)^{1/3} E(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi_m(\ell_0 + \epsilon, m)\, dm \\ & -A\int_{-\ell_0+\epsilon}^{\ell_0-\epsilon} (\ell_0 + \epsilon - m)^{1/3} E_m(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi(\ell_0 + \epsilon, m)\, dm,\end{aligned}$$

where $A = 1/2^{2/3}$.

Integrating by parts the first integral in (4.9), we get

$$\begin{aligned}(4.10)\qquad I_{\bar{B}_\epsilon} = {}& A(\ell_0 + \epsilon - m)^{1/3} E(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi(\ell_0 + \epsilon, m)\Big|_{-\ell_0+\epsilon}^{\ell_0-\epsilon} \\ & -\frac{A}{3}\int_{-\ell_0+\epsilon}^{\ell_0-\epsilon} (\ell_0 + \epsilon - m)^{-2/3} E(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi(\ell_0 + \epsilon, m)\, dm \\ & -2A\int_{-\ell_0+\epsilon}^{\ell_0-\epsilon} (\ell_0 + \epsilon - m)^{1/3} E_m(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi(\ell_0 + \epsilon, m)\, dm\end{aligned}$$

From equation (2.4) it follows that

$$R_m = -\frac{1}{3}(\ell - m)^{-2/3} E(\ell, m; \ell_0, -\ell_0) + (\ell - m)^{1/3} E_m(\ell, m; \ell_0, -\ell_0).$$

Substituting into (4.10), we obtain

$$(4.11)\qquad I_{\bar{B}_\epsilon} = A(\ell_0 + \epsilon - m)^{1/3} E(\ell_0 + \epsilon, m; \ell_0, -\ell_0)\psi(\ell_0 + \epsilon, m)\Big|_{-\ell_0+\epsilon}^{\ell_0-\epsilon}$$

$$-2A\int_{-\ell_0+\epsilon}^{\ell_0-\epsilon}[R_m(\ell_0+\epsilon,m;\ell_0,-\ell_0)+\frac{1}{6}\frac{R(\ell_0+\epsilon,m;\ell_0,-\ell_0)}{(\ell_0+\epsilon-m)}]\psi(\ell_0+\epsilon,m)\,dm.$$

Now, as $\epsilon\to 0$, the last integral tends to zero because, by Proposition 2.1, $R_m=-R/6(\ell-m)$ along the line $\ell=\ell_0$. On the other hand, the first term in (4.11), namely,

$$A(\ell_0+\epsilon-m)^{1/3}E(\ell_0+\epsilon,m;\ell_0,-\ell_0)\psi(\ell_0+\epsilon,m)\Big|_{-\ell_0+\epsilon}^{\ell_0-\epsilon}=$$
$$A(2\epsilon)^{1/3}E(\ell_0+\epsilon,\ell_0-\epsilon;\ell_0,-\ell_0)\psi(\ell_0+\epsilon,\ell_0-\epsilon)-$$
$$A(2\ell_0)^{1/3}E(\ell_0+\epsilon,-\ell_0+\epsilon;\ell_0,-\ell_0)\psi(\ell_0+\epsilon,-\ell_0+\epsilon).$$

tends to $-A(2\ell_0)^{1/3}E(\ell_0,-\ell_0;\ell_0,-\ell_0)\psi(\ell_0,-\ell_0)$, as $\epsilon\to 0$. Therefore,

$$\lim_{\epsilon\to 0}I_{\bar{B}_\epsilon}=\frac{-1}{2^{2/3}}\psi(\ell_0,-\ell_0). \tag{4.12}$$

because $E(\ell_0,-\ell_0;\ell_0,-\ell_0)=(2\ell_0)^{-1/3}$ and $A=1/2^{2/3}$.

4. *The integral along* $\bar{C}_\epsilon$. Denote by $I_{\bar{C}_\epsilon}$ the integral along the line $m=-\ell_0+\epsilon$ with $\ell_0+\epsilon<\ell<+\infty$. From formula (4.7) we now obtain

$$\begin{aligned}I_{\bar{C}_\epsilon}=A&\int_{\ell_0+\epsilon}^{+\infty}(\ell+\ell_0-\epsilon)^{1/3}E(\ell,-\ell_0+\epsilon;\ell_0,-\ell_0)\psi_\ell(\ell,-\ell_0+\epsilon)\,d\ell\\ -A&\int_{\ell_0+\epsilon}^{+\infty}(\ell+\ell_0-\epsilon)^{1/3}E_\ell(\ell,-\ell_0+\epsilon;\ell_0,-\ell_0)\psi(\ell,-\ell_0+\epsilon)\,d\ell,\end{aligned} \tag{4.13}$$

As before, integrate by parts the first integral, use the formula

$$R_\ell=\frac{1}{3}(\ell-m)^{-2/3}E(\ell,m;\ell_0,-\ell_0)+(\ell-m)^{1/3}E_\ell(\ell,m;\ell_0,-\ell_0).$$

to substitute for E_ℓ in the second integral, and rewrite (4.13) as

$$I_{\bar{C}_\epsilon}=-A(2\ell_0)^{1/3}E(\ell_0+\epsilon,-\ell_0,-\epsilon;\ell_0,-\ell_0)\psi(\ell_0+\epsilon,-\ell_0+\epsilon)$$

$$-2A\int_{\ell_0+\epsilon}^{+\infty}[R_\ell(\ell,-\ell_0+\epsilon;\ell_0,-\ell_0)-\frac{1}{6}\frac{R(\ell,-\ell_0+\epsilon;\ell_0,-\ell_0)}{(\ell+\ell_0-\epsilon)}]\psi(\ell,-\ell_0-\epsilon)\,d\ell.$$

As before, we obtain

$$\lim_{\epsilon\to 0}I_{\bar{C}_\epsilon}=\frac{-1}{2^{2/3}}\psi(\ell_0,-\ell_0). \tag{4.14}$$

5. *The integral along* $\bar{A}_\epsilon$. In this case, $m = \ell_0 - \epsilon$, ℓ varies from $+\infty$ to $\ell_0 + \epsilon$, and the integral to be considered is

$$I_{\bar{A}_\epsilon} = -A \int_{\ell_0+\epsilon}^{+\infty} (\ell - \ell_0 + \epsilon)^{1/3} E(\ell, \ell_0 - \epsilon; \ell_0, -\ell_0) \psi_\ell(\ell, \ell_0 - \epsilon)\, d\ell$$
$$+ A \int_{\ell_0+\epsilon}^{+\infty} (\ell - \ell_0 + \epsilon)^{1/3} E_\ell(\ell, \ell_0 - \epsilon; \ell_0, -\ell_0) \psi(\ell, \ell_0 - \epsilon)\, d\ell. \tag{4.15}$$

Our aim is to show that

$$\lim_{\epsilon \to 0} I_{\bar{A}_\epsilon} = 0. \tag{4.16}$$

Once this is done, then by adding (4.12), (4.14), and (4.16), we obtain (4.8) and the theorem will be proved.

As we did in the previous two cases, we integrate by parts the first integral in (4.15) and get

$$I_{\bar{A}_\epsilon} = A(2\epsilon)^{1/3} E(\ell_0 + \epsilon, \ell_0 - \epsilon; \ell_0, -\ell_0) \psi(\ell_0 + \epsilon, \ell_0 - \epsilon)$$
$$+ \frac{A}{3} \int_{\ell_0+\epsilon}^{+\infty} (\ell - \ell_0 + \epsilon)^{-2/3} E(\ell, \ell_0 - \epsilon; \ell_0, -\ell_0) \psi(\ell, \ell_0 - \epsilon)\, d\ell$$
$$+ 2A \int_{\ell_0+\epsilon}^{+\infty} (\ell - \ell_0 + \epsilon)^{1/3} E_\ell(\ell, \ell_0 - \epsilon; \ell_0, -\ell_0) \psi(\ell, \ell_0 - \epsilon)\, d\ell. \tag{4.17}$$

In order to prove (4.16) we now must take into account the asymptotic behavior of both $E(\ell, m; \ell_0, -\ell_0)$ and $E_\ell(\ell, m; \ell_0, -\ell_0)$ for m near ℓ_0 and, more specifically, the behavior of the hypergeometric function $F(1/6, 1/6; 1; \zeta)$ and its derivative

$$F'(\frac{1}{6}, \frac{1}{6}; 1; \zeta) = \frac{1}{36} F(\frac{7}{6}, \frac{7}{6}; 2; \zeta), \tag{4.18}$$

as $\zeta \to \infty$.

To see this, we start by expressing (4.17) in terms of $F(1/6, 1/6; 1; \zeta)$ and its derivative. In order to simplify notations we write from now on, $F(\zeta) = F(1/6, 1/6; 1; \zeta)$ and $G(\zeta) = F(7/6, 7/6; 2; \zeta)$. In the case we are considering $m = \ell_0 - \epsilon$ so, in view of formula (2.7), we set

$$\zeta = \frac{(2\ell_0 - \epsilon)(\ell - \ell_0)}{-\epsilon(\ell + \ell_0)} = \frac{\sigma(\epsilon)}{-\epsilon} \qquad \text{with} \qquad \sigma(\epsilon) = \frac{(2\ell_0 - \epsilon)(\ell - \ell_0)}{(\ell + \ell_0)}.$$

Also, recalling formula (2.6), we have

$$E(\ell,\ell_0-\epsilon;\ell_0,-\ell_0)=\epsilon^{-1/6}(\ell+\ell_0)^{-1/6}F(\frac{\sigma(\epsilon)}{-\epsilon}), \tag{4.19}$$

and so

$$E_\ell(\ell,\ell_0-\epsilon;\ell_0,-\ell_0)= \tag{4.20}$$
$$-\frac{\epsilon^{-1/6}}{6}(\ell+\ell_0)^{-7/6}F(\frac{\sigma(\epsilon)}{-\epsilon})-\frac{\epsilon^{-7/6}\ell_0(2\ell_0-\epsilon)}{18}(\ell+\ell_0)^{-13/6}G(\frac{\sigma(\epsilon)}{-\epsilon}),$$

in view of (4.18) and taking into account that

$$\frac{d\zeta}{d\ell}=\frac{2\ell_0(2\ell_0-\epsilon)}{-\epsilon(\ell+\ell_0)^2}.$$

By substituting (4.19) and (4.20) into (4.17) and combining the resulting integrals we obtain

$$I_{\bar{A}_\epsilon}=A(2\epsilon)^{1/3}(2\ell_0+\epsilon)^{-1/6}\epsilon^{-1/6}F(\frac{\epsilon-2\ell_0}{2\ell_0+\epsilon})\psi(\ell_0+\epsilon,\ell_0-\epsilon) \tag{4.21}$$
$$+\frac{A}{3}(2\ell_0-\epsilon)^{5/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}(\frac{\sigma(\epsilon)}{\epsilon})^{1/6}F(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell$$
$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}(\frac{\sigma(\epsilon)}{\epsilon})^{7/6}G(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell.$$

As $\epsilon\to 0$, the first term on the right-hand side of (4.21) clearly tends to zero. We must prove that the difference of the two integrals in (4.21) also tends to zero as $\epsilon\to 0$. For this, according to formulas (4.1), (4.2), and (4.3), we have

$$(\frac{\sigma(\epsilon)}{\epsilon})^{1/6}F(\frac{\sigma(\epsilon)}{-\epsilon})=\log(\frac{\sigma(\epsilon)}{\epsilon})u(\frac{\sigma(\epsilon)}{-\epsilon})+v(\frac{\sigma(\epsilon)}{-\epsilon})$$

and

$$(\frac{\sigma(\epsilon)}{\epsilon})^{7/6}G(\frac{\sigma(\epsilon)}{-\epsilon})=\log(\frac{\sigma(\epsilon)}{\epsilon})U(\frac{\sigma(\epsilon)}{-\epsilon})+V(\frac{\sigma(\epsilon)}{-\epsilon}).$$

By using these two expressions we combine the two integrals in (4.21) and write them as a sum $I^{(1)}_{\bar{A}_\epsilon}+I^{(2)}_{\bar{A}_\epsilon}$ - $\log\epsilon\cdot I^{(3)}_{\bar{A}_\epsilon}$, where

$$I^{(1)}_{\bar{A}_\epsilon}= \tag{4.22}$$

$$\frac{A}{3}(2\ell_0-\epsilon)^{5/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}\log(\sigma(\epsilon))u(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell$$
$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}\log(\sigma(\epsilon))U(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell,$$

$$I^{(2)}_{\bar{A}_\epsilon}= \tag{4.23}$$
$$\frac{A}{3}(2\ell_0-\epsilon)^{5/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}v(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell$$
$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}V(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell,$$

and

$$I^{(3)}_{\bar{A}_\epsilon}= \tag{4.24}$$
$$\frac{A}{3}(2\ell_0-\epsilon)^{5/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}u(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell$$
$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}U(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell.$$

Lemma 4.1. *Each of the integrals* $I^{(i)}_{\bar{A}_\epsilon}, 1\le i\le 3,$ *is* $O(\epsilon^{1/6})$.

Proof of lemma. Consider first the integral $I^{(1)}_{\bar{A}_\epsilon}$. Since the series in (4.2) converge for large values of $|\zeta|$, we may write that

$$u(\frac{\sigma(\epsilon)}{-\epsilon})=u_0+\frac{\epsilon}{\sigma(\epsilon)}\tilde{u}(\frac{\sigma(\epsilon)}{-\epsilon})\qquad\text{and}\qquad U(\frac{\sigma(\epsilon)}{-\epsilon})=U_0+\frac{\epsilon}{\sigma(\epsilon)}\tilde{U}(\frac{\sigma(\epsilon)}{-\epsilon}),$$

where $\tilde{u}$ and $\tilde{U}$ are bounded functions for small values of ϵ.

By substituting these two expressions into (4.22), rewrite it as a sum of three terms:

$$I^{(1)}_{\bar{A}_\epsilon}= \tag{4.25}$$
$$\{\frac{A}{3}(2\ell_0-\epsilon)^{5/6}u_0\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}\log(\sigma(\epsilon))\psi(\ell,\ell_0-\epsilon)\,d\ell$$
$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}U_0\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}\log(\sigma(\epsilon))\psi(\ell,\ell_0-\epsilon)\,d\ell\}$$

$$+\frac{A}{3}(2\ell_0-\epsilon)^{5/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{-2/3}}{(\ell-\ell_0)^{1/6}(\ell+\ell_0)}\log(\sigma(\epsilon))\frac{\epsilon}{\sigma(\epsilon)}\tilde{u}(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell$$

$$-\frac{A}{9}\ell_0(2\ell_0-\epsilon)^{-1/6}\int_{\ell_0+\epsilon}^{+\infty}\frac{(\ell-\ell_0+\epsilon)^{1/3}}{(\ell-\ell_0)^{7/6}(\ell+\ell_0)}\log(\sigma(\epsilon))\frac{\epsilon}{\sigma(\epsilon)}\tilde{U}(\frac{\sigma(\epsilon)}{-\epsilon})\psi(\ell,\ell_0-\epsilon)\,d\ell.$$

The first term (inside the brackets) in (4.25) tends to 0 as $\epsilon\to 0$. Indeed, its limit as $\epsilon\to 0$ is

$$[\frac{A(2\ell_0)^{5/6}u_0}{3}-\frac{A\ell_0(2\ell_0)^{-1/6}U_0}{9}]\int_{\ell_0}^{\infty}\frac{(\ell-\ell_0)^{-5/6}}{\ell+\ell_0}\log(\frac{2\ell_0(\ell-\ell_0)}{\ell+\ell_0})\psi(\ell,\ell_0)\,d\ell.$$

The integral on the right-hand side is finite. On the other hand,

$$u_0=\frac{1}{\Gamma(1/6)\Gamma(5/6)}\quad\text{and}\quad U_0=\frac{1}{\Gamma(5/6)\Gamma(7/6)}=\frac{6}{\Gamma(1/6)\Gamma(5/6)},$$

are the constant terms of the series $u(\zeta)$ and $U(\zeta)$ given by (4.2). It is a matter of verification that the quantity inside the brackets is equal zero.

Next, consider the second term in (4.25). After the change of variables $\ell-\ell_0=t\epsilon$, one can see that the absolute value of that term is bounded above by

$$C\epsilon^{1/6}\int_1^{+\infty}(t+1)^{-2/3}t^{-7/6}\log t\,dt,$$

with C a constant independent of ϵ. With a similar calculation, one can show that the third term in (4.25) is bounded above by

$$C\epsilon^{1/6}\int_1^{+\infty}(t+1)^{1/3}t^{-13/6}\log t\,dt,$$

with C another constant. Therefore, $I^{(1)}_{\bar{A}_\epsilon}=O(\epsilon^{1/6})$.

The expression (4.24) for $I^{(3)}_{\bar{A}_\epsilon}$, similar to that of $I^{(1)}_{\bar{A}_\epsilon}$, is even simpler because of the absence of the factor $\log(\sigma(\epsilon))$. Thus, with a similar proof we also conclude that $I^{(3)}_{\bar{A}_\epsilon}=O(\epsilon^{1/6})$. Finally, the integral $I^{(2)}_{\bar{A}_\epsilon}$ given by (4.23) is analogous to $I^{(3)}_{\bar{A}_\epsilon}$ with u and U replaced by v and V the power series defined by (4.3). The proof of the lemma is then complete. □

Recalling the expression (4.22) for $I_{\bar{A}_\epsilon}$ one sees that Lemma 4.1 implies (4.16) which completes the proof of the theorem. □

5 Fundamental Solutions in the region D_{II}

In the same manner as we did in the previous sections, define the distribution

$$E_{II}(x, y; 0, b) = \begin{cases} \dfrac{1}{2^{1/3}} E(x, y; 0, b) & \text{in } D_{II} \\ \\ 0 & \text{elsewhere,} \end{cases} \tag{5.1}$$

whose support is the closure of the region D_{II}. Our aim is to prove the following result.

Theorem 5.1. *$E_{II}(x, y; 0, b)$ is a fundamental solution for the Tricomi operator relative to the point* $(0, b)$.

Proof. The proof is, with few modifications, analogous to that of Theorem 4.1. As before, it suffices to show that

$$\langle E, \mathcal{T}\varphi \rangle = \lim_{\epsilon \to 0} \iint_{D_\epsilon} E\mathcal{T}\varphi \, dx \, dy = 2^{1/3} \varphi(0, b), \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2), \tag{5.2}$$

where the domain of integration D_ϵ is defined as follows. Let D be an open

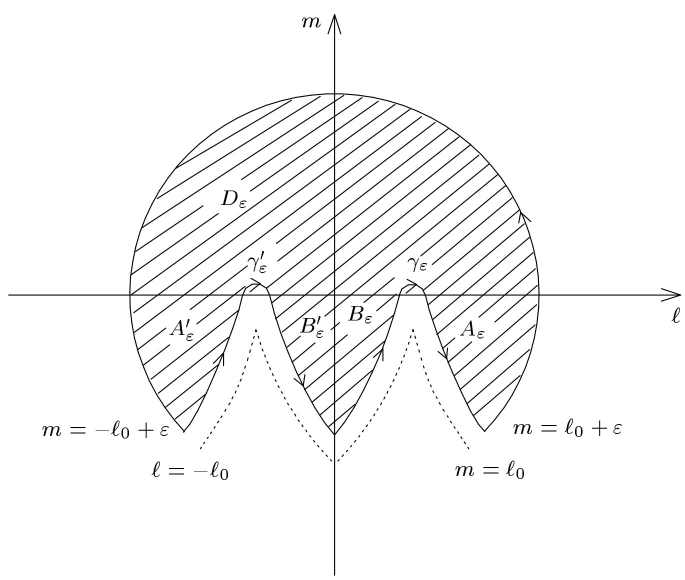

Figure 5

disk centered at the origin and with radius sufficiently large so that it contains the points $(-a,0)$, $(a,0)$, and $(0,b)$ and the the support of φ. Let $D_{II,\epsilon}$ be the region obtained from D_{II} by shifting it by ϵ along the characteristics and by removing two small half disks centered at the points $(-a,0)$, $(a,0)$, as shown in the Figure 5. D_ϵ is then the intersection of D and $D_{II,\epsilon}$.

By virtue of the Green's formula (4.6), the double integral in the expression (5.2) is to be replaced by a contour integral along the oriented paths A'_ϵ, γ'_ϵ, B'_ϵ, B_ϵ, γ_ϵ, and A_ϵ.

The integration along A_ϵ is similar to the one along $\bar{A}_\epsilon$ calculated in Section 4 and one can see that its limit, as $\epsilon \to 0$, is zero. The same is true for the integral along A'_ϵ.

The integral along B_ϵ, similar to the integral along $\bar{B}_\epsilon$ (Section 4), tends to $\psi(\ell_0,-\ell_0)/2^{2/3}$, as $\epsilon \to 0$. The same thing happens with the integral along B'_ϵ. The sum of these two limits is then $2^{1/3}\psi(\ell_0,-\ell_0)$ which is equal to the right-hand side of (5.2). Recall that $\psi(\ell,m) = \varphi((\ell+m)/6, -((\ell-m)/4)^{2/3})$.

To complete the proof one has to show that the limits, as $\epsilon \to 0$, of the integrals along γ'_ϵ and γ_ϵ are both zero. Since these two contours lie in the elliptic region of the Tricomi operator, it is more convenient to use the *reduced elliptic* form of the Tricomi operator, namely,

$$\mathcal{T}_e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} + \frac{1}{3s}\frac{\partial}{\partial s}. \tag{5.3}$$

which one obtains from equation (1.1) via the change of variables $x = x$ and $s = 2y^{3/2}/3$, and the corresponding Green's formula in the variables x and s. It is then a matter of verification, which we leave to the reader, that along both contours the integrands remain bounded and consequently both integrals along γ'_ϵ and γ_ϵ tend to zero. The proof is then complete. $\square$

We observe that by virtue of Proposition 2.2, the fundamental solution $E_{II}(x,y;0,b)$ is a complex valued in the upper half plane ($y > 0$) plus the region in the lower half plane outside the reflected characteristics and it is real valued in the triangle with vertices $(-a,0)$, $(a,0)$, and $(0,b)$. Since the Tricomi operator has real coefficients, the complex conjugate $\bar{E}_{II}(x,y;0,b)$ as well as the real part of $E_{II}(x,y;0,b)$ are fundamental solutions for $\mathcal{T}$.

Contrary to what happened in Corollary 3.1 of Section 4 where we showed how the limit of $E_I(x,y;0,b)$, as $(0,b) \to (0,0)$, tends to the fundamental solution $F_-(x,y)$ given by formulas (1.5) and (1.6), neither of the fundamental solutions just obtained tend to the fundamental solution $F_+(x,y)$ given

by formulas (1.3) and (1.4). For this to happen we need a particular linear combination of E_{II} and $\bar{E}_{II}$. Let λ and μ be such that

$$\begin{cases} \lambda e^{\frac{i\pi}{6}} + \mu e^{-\frac{i\pi}{6}} &= -1/3^{1/2} \\ \lambda + \mu &= \ \ 1. \end{cases}$$

Then, the following result holds:

Corollary 5.1. *Let* $E^{\sharp}_{II} = \lambda E_{II} + \mu \bar{E}_{II}$. *Then* $E^{\sharp}_{II}$ *is a fundamental solution relative to* $(0, b)$ *that converges in the sense of distributions to* $F_+(x, y)$, *as* $(0, b) \to (0, 0)$.

References

[1] S. Agmon, *Boundary Value Problems for Equations of Mixed Type*, Atti del Convegno Internazionale sulle Equazioni alle Derivate Parziali, 54-68, Triestre (1954).

[2] J. Barros-Neto and I. M. Gelfand, *Fundamental Solutions for the Tricomi Operator*, Duke Math. Journal, **98**, no. 3, (1999), 465-483.

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I, Interscience Publishers, New York, 1953.

[4] Darboux, *Leçons sur la Théorie des Surfaces*, vol. II, Gauthier-Villars, Paris, 1914.

[5] A. Erdély, *Higher Transcendental Functions*, The Bateman Manuscript Project, Vol. 1, 2, 3, McGraw-Hill, 1953.

[6] K. O. Friedrichs, *Symmetric Positive Linear Differential Equations*, Comm. Pure Appl. Math., **11**, 333-418 (1958).

[7] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, 1964.

[8] P. Germain and R. Bader, *Solutions Élémentaires de Certaines Équations aux Dérivées Partielles du Type Mixte*, Bull. Soc. Math. de France, **81**, 1953, 145-174 (1953).

[9] L. D. Landau and E .M. Lifshitz, *Fluid Mechanics; Course of Theoretical Physics*, Vol. IV, Pergamon Press, London, 1959.

[10] J. Leray, *La Solution Unitaire d'un Opérateur Différentiel Linéaire (Problème de Cauchy, II)*, Bull. Soc. Math. France, **86**, fasc. I, 75–96 (1958).

[11] J. Leray, *Un Prolongement de la Transformation de Laplace qui Transforme la Solution Unitaire d'un Opérateur Hyperbolique en sa Solution Élémentaire (Problème de Cauchy, IV)*, Bull. Soc. Math. France, **90**, fasc. I, 39–156 (1962).

[12] C. S. Morawetz, *A weak solution for a system of equations of elliptic-hyperbolic type*, Comm. Pure Appl. Math. **11** (1958), 315–331.

[13] L. Schwartz, *Théorie des Distributions,* Hermann, Paris, 1950–51.

[14] F. Tricomi, *Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto,* Rendiconti, Atti dell'Accademia Nazionali dei Lincei, Serie 5, 14, (1923), 134-247.